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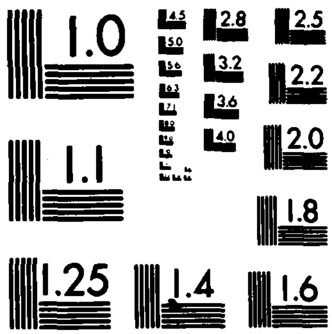

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BY

FRED W. HUFFER

TECHNICAL REPORT NO. 378
AUGUST 5, 1986

PREPARED UNDER CONTRACT
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SLEPIAN'S INEQUALITY VIA THE
CENTRAL LIMIT THEOREM

By

Fred W. Huffer

We give a proof of Slepian's (1962) inequality which does not rely on Plackett's identity or geometric arguments. The proof uses a partial ordering of distributions which is preserved under convolutions and scale transformations. Slepian's inequality may be formulated in terms of such a partial ordering. The properties of this partial ordering allow us to obtain results for the multivariate normal distribution by using the central limit theorem.

Tchen (1980) also noted the preservation under convolution property and from this obtained Slepian's inequality in the bivariate case. Further information and references concerning partial orderings of probability distributions may be found in Eaton (1982) or Chapter 1 of Stoyan (1983).

Let $\mathcal{F}$ be a collection of bounded continuous real-valued functions defined on $\mathbb{R}^k$. Suppose that $\mathcal{F}$ is invariant under both translations and scaling, that is, for any $b \in \mathbb{R}^k$, $c \geq 0$ and $f \in \mathcal{F}$, the function g defined by $g(x) = f(cx+b)$ also belongs to $\mathcal{F}$. Define $X << Y$ if X and Y are random vectors satisfying $Ef(X) \leq Ef(Y)$ for all $f \in \mathcal{F}$.

Our object will be to prove inequalities concerning multivariate normal distributions. Let Σ and Λ be any $k \times k$ covariance matrices. Let X^* and Y^* be k-dimensional normal random vectors with $EX^* = EY^* = 0$,

$\text{cov } X^* = \Sigma$, $\text{cov } Y^* = \Lambda$. We wish to determine if $X^* << Y^*$ in which case we also say that $\Sigma << \Lambda$. The following result can sometimes be used to make this determination.

Proposition 1: Let X^* and Y^* be as given above. Suppose the random vectors X and Y satisfy $EX = EY = 0$, $\text{cov } X = \Sigma$, $\text{cov } Y = \Lambda$. If $X << Y$, then $X^* << Y^*$.

In our application X and Y will be chosen to have simple discrete distributions so that the relationship $X << Y$ is easy to verify.

Proof: First note that

(a) If U,V,W are independent with $U << V$, then $U+W << V+W$.

This follows by conditioning on the value of W and using the translation invariance of $\mathcal{F}$. Let $X_1,X_2,X_3,\ldots$ be i.i.d. copies of X and $Y_1,Y_2,Y_3,\ldots$ be i.i.d. copies of Y. Since $X << Y$, using (a) twice in succession gives $X_1+X_2 << Y_1+X_2 << Y_1+Y_2$. By induction we obtain $X_1+X_2 +\cdots+ X_n << Y_1+Y_2 +\cdots+ Y_n$. By the scale invariance of $\mathcal{F}$,

(b) $U << V$ implies $cU << cV$ for all $c \geq 0$.

Thus $n^{-1/2}(X_1+X_2+\cdots+X_n) << n^{-1/2}(Y_1+Y_2+\cdots+Y_n)$. Now let $n \to \infty$ and use the Central Limit Theorem to obtain $X^* << Y^*$.

The next proposition is useful in extending the ordering $<<$ to a broader class of covariance matrices.

Proposition 2: Suppose Λ_1 and Λ_2 are any $k \times k$ covariance matrices satisfying $\Lambda_1 << \Lambda_2$. Define $\Gamma = \Lambda_2 - \Lambda_1$. Choose $t > 0$. If Σ and $\Sigma + t\Gamma$ are both nonsingular covariance matrices, then $\Sigma << \Sigma + t\Gamma$.

Proof: Let Φ_1, Φ_2, Φ_3 denote $k \times k$ covariance matrices. In terms of covariance matrices (a) and (b) become:

(c) $$\Phi_1 << \Phi_2 \text{ implies } \Phi_1 + \Phi_3 << \Phi_2 + \Phi_3 .$$

(d) $$\Phi_1 << \Phi_2 \text{ implies } c\Phi_1 << c\Phi_2 \text{ for all } c \geq 0 .$$

These are used implicitly in the following argument. Choose Δ small enough so that both $(\Sigma - \varepsilon\Lambda_1)$ and $(\Sigma + t\Gamma - \varepsilon\Lambda_1)$ are positive definite whenever $0 \leq \varepsilon \leq \Delta$. By taking convex combinations of these matrices we find that $(\Sigma + s\Gamma - \varepsilon\Lambda_1)$ is positive definite (and therefore a covariance matrix) when $0 \leq s \leq t$ and $0 \leq \varepsilon \leq \Delta$. Now

$$\Sigma + s\Gamma = (\Sigma + s\Gamma - \varepsilon\Lambda_1) + \varepsilon\Lambda_1 << (\Sigma + s\Gamma - \varepsilon\Lambda_1) + \varepsilon\Lambda_2 = \Sigma + (s+\varepsilon)\Gamma .$$

Here we have used $\Lambda_1 << \Lambda_2$. Thus $\Sigma + s\Gamma << \Sigma + (s+\varepsilon)\Gamma$ for $0 \leq s \leq t$ and $0 \leq \varepsilon \leq \Delta$. Since Δ does not depend on s, it is clear that $\Sigma << \Sigma + t\Gamma$ as desired.

For $x = (x_1, x_2, \ldots, x_k)$ and $y = (y_1, y_2, \ldots, y_k)$ define $x \vee y = (x_1 \vee y_1, x_2 \vee y_2, \ldots, x_k \vee y_k)$ and $x \wedge y = (x_1 \wedge y_1, x_2 \wedge y_2, \ldots, x_k \wedge y_k)$ where $\vee$ and $\wedge$ denote the maximum and minimum respectively. A function f defined on $\mathbb{R}^k$ is called L-superadditive if $f(x) + f(y) \leq f(x \vee y) + f(x \wedge y)$

for all x and y. This condition was introduced by Lorentz (1953) who also showed that when f has continuous second partial derivatives, f is L-superadditive if and only if

$$\frac{\partial^2 f(x)}{\partial x_i \partial x_j} \geq 0 \quad \text{for all } x \text{ and all } i \neq j .$$

See Marshall and Olkin (1979) for further information on L-superadditivity.

<u>Proposition 3</u>: Let $\mathcal{F}$ be the class of bounded, continuous, L-superadditive functions on $\mathbb{R}^k$. Suppose $\Sigma = (\sigma_{ij})$ and $\Pi = (\pi_{ij})$ are $k \times k$ non-singular covariance matrices. If $\sigma_{ii} = \pi_{ii}$ for all i and $\sigma_{ij} \leq \pi_{ij}$ for all $i \neq j$, then $\Sigma << \Pi$.

This result is very similar to Proposition 1 of Joag-dev, Perlman and Pitt (1983) and it easily implies Slepian's inequality as given in Slepian (1962). The argument needed to obtain Slepian's inequality is basically the same as that in Corollary 1 of Joag-dev, et al.

<u>Proof</u>: Let $\alpha, \beta, \theta, \phi$ be k-dimensional vectors defined by

$$\alpha_p = 1,\ \alpha_q = -1,\ \alpha_i = 0 \quad \text{for } i \neq p \text{ or } q ,$$

$$\beta_p = -1,\ \beta_q = 1,\ \beta_i = 0 \quad \text{for } i \neq p \text{ or } q ,$$

$$\theta_p = \theta_q = 1,\ \theta_i = 0 \quad \text{for } i \neq p \text{ or } q ,$$

$$\phi_p = \phi_q = -1,\ \phi_i = 0 \quad \text{for } i \neq p \text{ or } q ,$$

Define the random vectors X and Y by

$$P\{X=\alpha\} = P\{X=\beta\} = \frac{1}{2},\ P\{Y=\theta\} = P\{Y=\phi\} = \frac{1}{2} .$$

Since $\alpha \vee \beta = \theta$ and $\alpha \wedge \beta = \phi$, it is clear that $X \ll Y$. Now applying Proposition 1 leads to a corresponding ordering between normal random vectors with covariance matrices

$$\operatorname{cov} X = S^{pq} \quad \text{and} \quad \operatorname{cov} Y = T^{pq}$$

where the entries of S^{pq} and T^{pq} are given by $S^{pq}_{pp} = S^{pq}_{qq} = 1$, $S^{pq}_{pq} = S^{pq}_{qp} = -1$ and $S^{pq}_{ij} = 0$ otherwise, $T^{pq}_{pp} = T^{pq}_{qq} = T^{pq}_{pq} = \Gamma^{pq}_{qp} = 1$ and $T^{pq}_{ij} = 0$ otherwise. Since p and q are arbitrary, we have shown $S^{pq} \ll T^{pq}$ for all $p \neq q$. Now we can use (c) and (d) to deduce that $\Lambda_1 \ll \Lambda_2$ where

$$\Lambda_1 = \sum_{\substack{p,q \\ p<q}} b_{pq} S^{pq},\ \Lambda_2 = \sum_{\substack{p,q \\ p<q}} b_{pq} T^{pq}$$

and b_{pq} are arbitrary nonnegative numbers. Choose $b_{ij} = (\pi_{ij} - \sigma_{ij})/2$ for all $i < j$. Now using Proposition 2 with $\Gamma = \Lambda_2 - \Lambda_1 = \Pi - \Sigma$ completes the proof.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We define a partial ordering of distributions which is preserved under convolutions and scale transformations. Some properties of this partial ordering are developed and then used to give a new argument for Slepian's (1962) inequality.

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